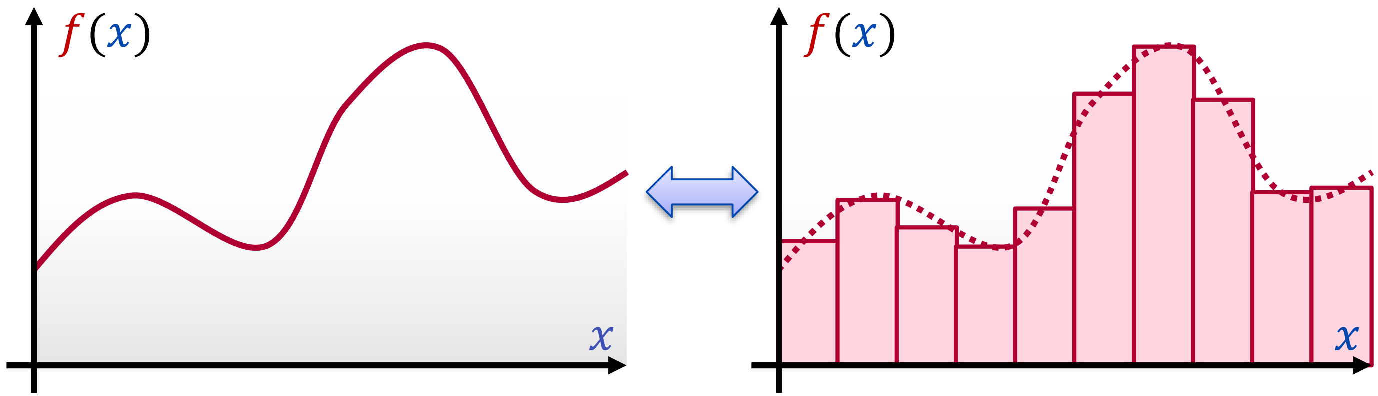
Modelling 1

SUMMER TERM 2020



LECTURE 2

Vector Spaces

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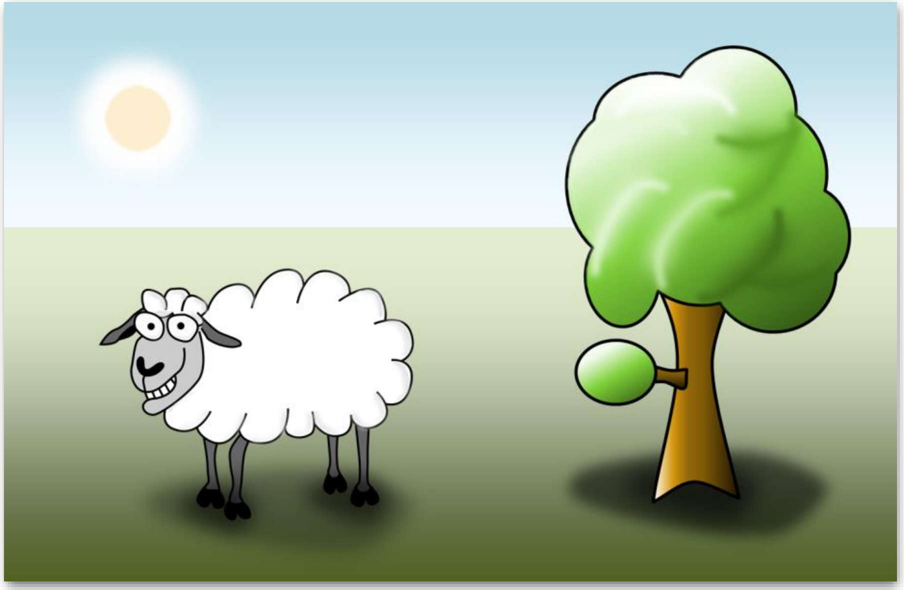
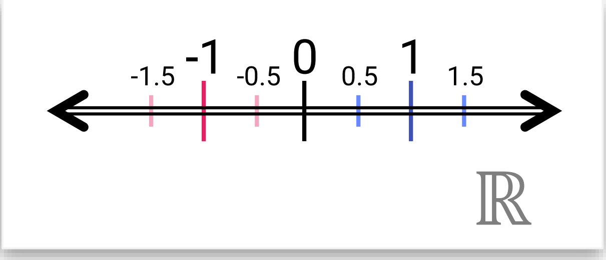
Euclidean Geometry & Vector Spaces



Development

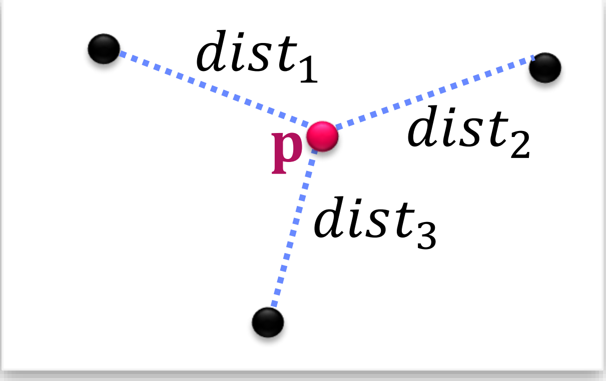
One-dimensional quantities

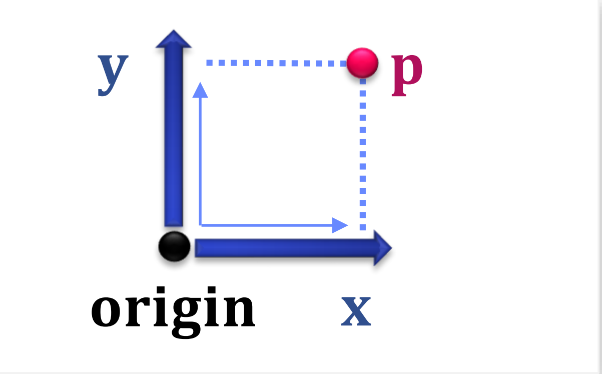
- Real numbers
- Continuous straight line



Higher dimensions

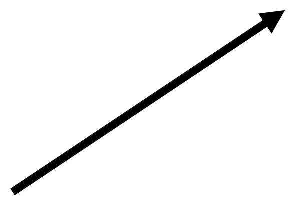
- Distances to landmarks
  - Non-linear (hard to calculate)
- Cartesian coordinates
  - Projection on coordinate axes
  - Linear structure
  - Convenient & fully understood





Vectors

Vectors

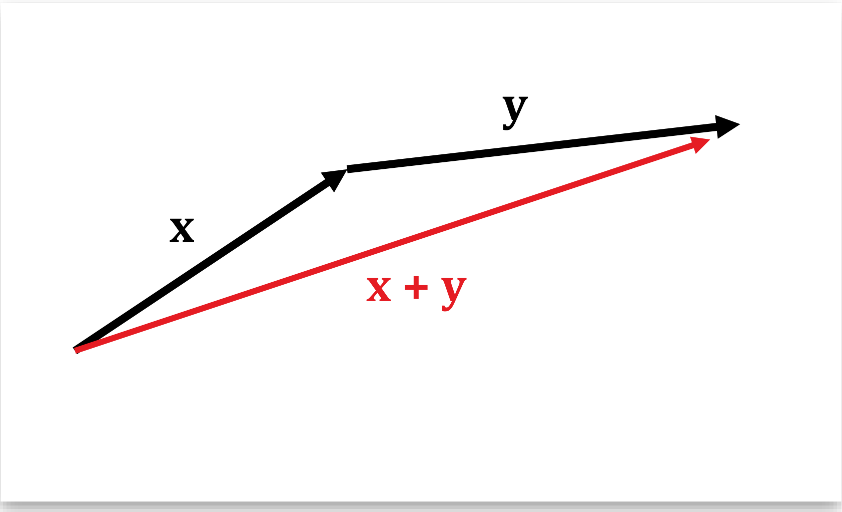


$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

Geometry:
vectors are arrows in space

Algebra:
arrays of numbers

Vector Addition



Adding Vectors:
Concatenation

$$\mathbf{x} + \mathbf{y} = \begin{pmatrix} x_1 + y_1 \\ x_2 + y_2 \\ x_3 + y_3 \end{pmatrix}$$

Algebra:
adding numbers

Structure: Abelian Group

Group

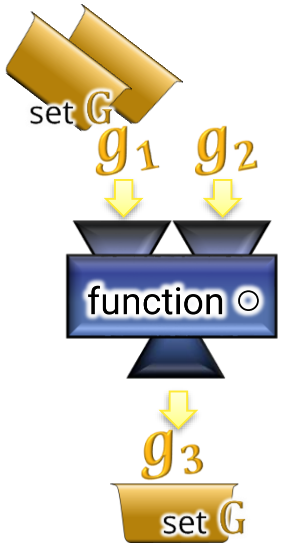
C++ pseudo-code:

group

```
template <set T, operator ○>
T operator"○"(T, T)
```

Axioms: closed operation, associative, neutral element, inverse

schematic:

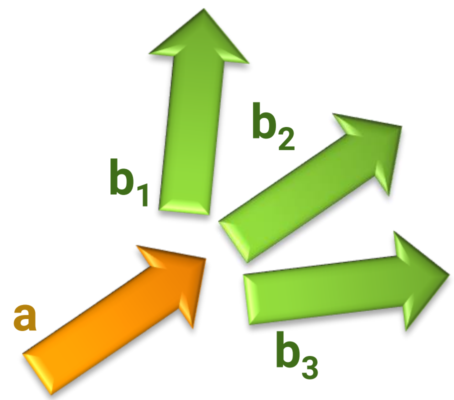


Group Axioms

- Closed: Set G, closed operation " $\circ$ ": $G, G \rightarrow G$
- Associative: $(g_1 \circ g_2) \circ g_3 = g_1 \circ (g_2 \circ g_3)$
- There is a neutral element $id \in G$: $id \circ g = g \circ id = g$
- For each $g \in G$ there is an inverse $g^{-1} \in G$:

$$g \circ g^{-1} = g^{-1} \circ g = id$$

What do the axioms mean?

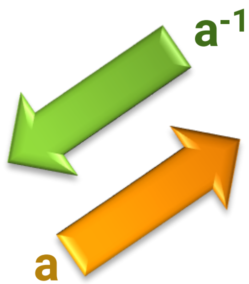


closed operation

all operations always possible

$$\forall a, b \in G\colon\ a \circ b \in G$$

What do the axioms mean?



Neutral element

(unique) null operation

$\forall a \in G\colon\ a \circ id = a$

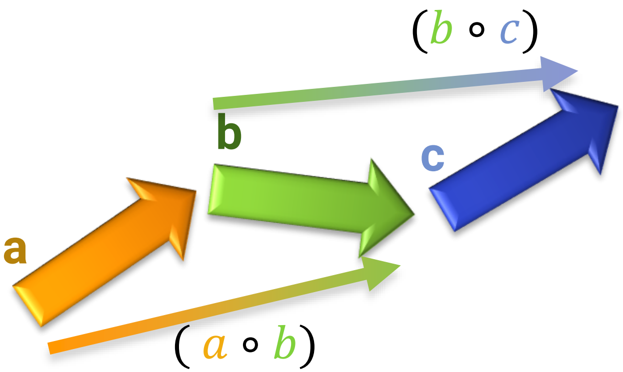
Inverse

all operations reversible

$\forall a \in G\colon\ a \circ a^{-1} = id$

no information loss!

What do the axioms mean?

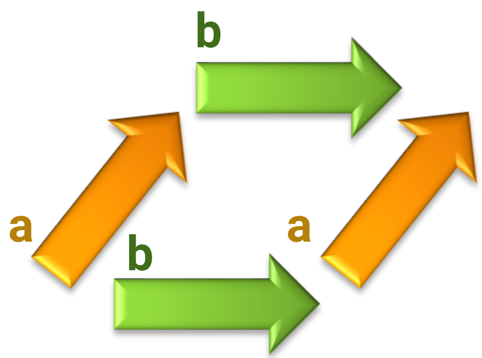


associativity

effect "adds up": operations can be summarized / grouped together consistently

$$\forall a, b, c \in G: (a \circ b) \circ c = a \circ (b \circ c)$$

What do the axioms mean?

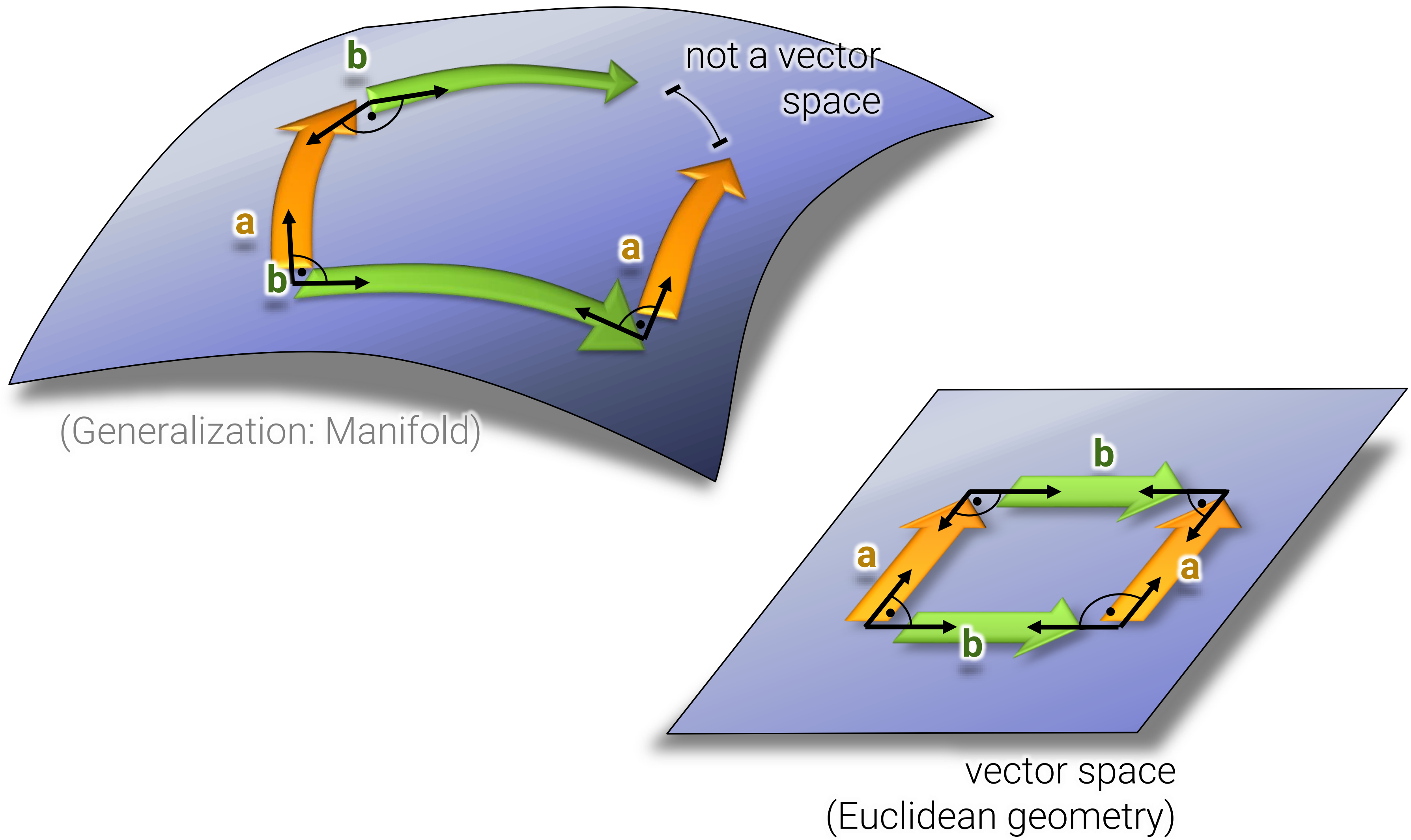


commutativity

intuition: grid / flat structure

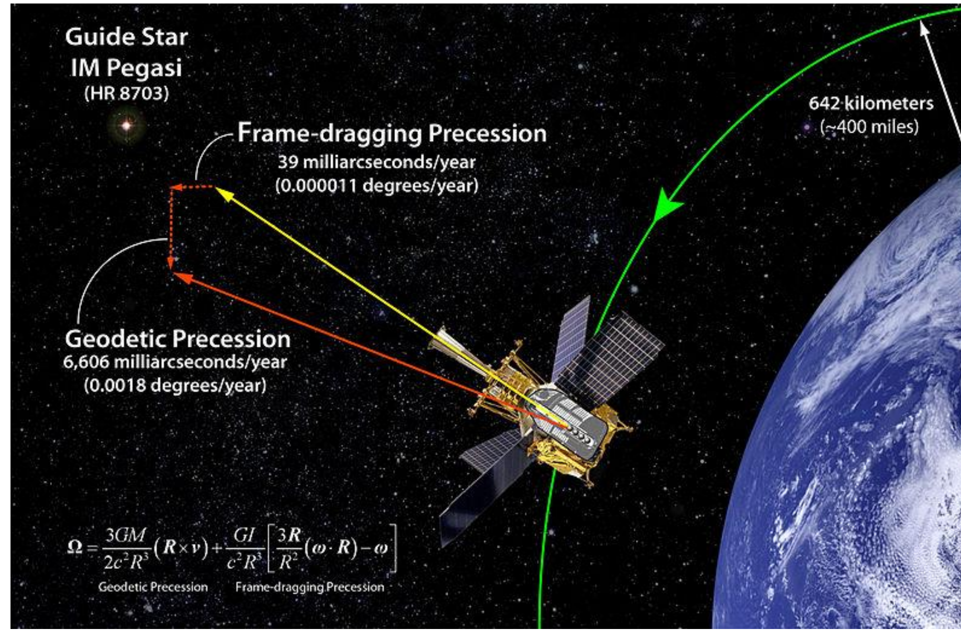
$\forall a, b \in G: \; a \circ b = \; b \circ a$

Euclidean Space is not Curved



Beyond Middle-World

[NASA]

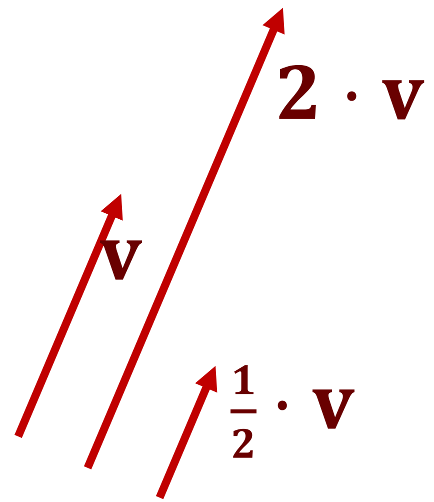


[NASA]

http://en.wikipedia.org/wiki/Gravity_Probe_B

Back to the Vectors…

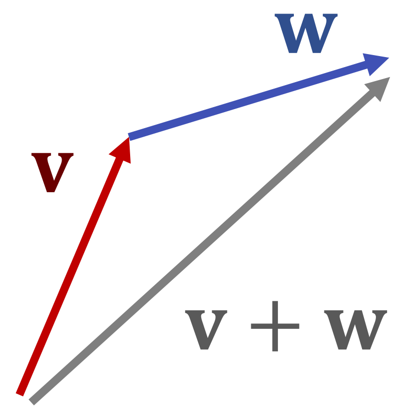
Vector Operations



scaling vectors

vector-scalar product

$\lambda \cdot \mathbf{v} \quad (\lambda \in \mathbb{R},\ \mathbf{v} \in V)$



adding vectors

vector-addition

$\mathbf{v} + \mathbf{w} \quad (\mathbf{v}, \mathbf{w} \in V)$

Structure: Vector Space

Vector Spaces

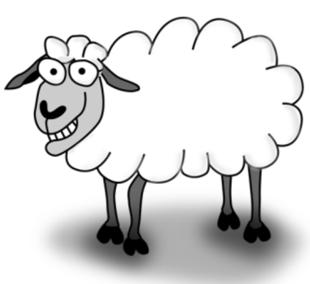
Vector space:

- Set of vectors $\mathbb{V}$
- Based on field $\mathbb{F}$ (usually $\mathbb{F} = \mathbb{R}$)
- Two operations:
  - Adding vectors $\mathbf{u} = \mathbf{v} + \mathbf{w}$ $(\mathbf{u}, \mathbf{v}, \mathbf{w} \in V)$
  - Scaling vectors $\mathbf{w} = \lambda\mathbf{v}$ $(\mathbf{u} \in V, \lambda \in \mathbb{F})$

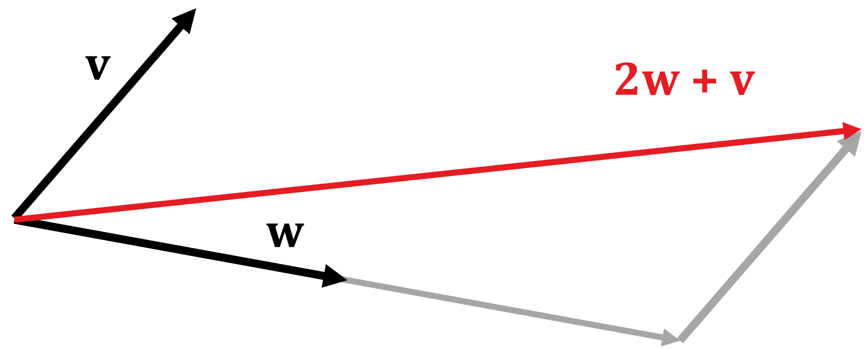
Vector Spaces

Vector space axioms:

- Vector addition – Abelian group:
  - $\forall \mathbf{u}, \mathbf{v}, \mathbf{w} \in V$: $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$
  - $\forall \mathbf{u}, \mathbf{v} \in V$: $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$
  - $\exists \mathbf{0} \in V: \forall \mathbf{v} \in V$: $\mathbf{v} + \mathbf{0} = \mathbf{v}$
  - $\forall \mathbf{v} \in V: \exists "{-\mathbf{v}}" \in V$: $\mathbf{v} + (-\mathbf{v}) = \mathbf{0}$
- Compatibility with scalar multiplication:
  - $\forall \mathbf{v} \in V, \lambda, \mu \in F$: $\lambda(\mu\mathbf{v}) = \lambda\mu(\mathbf{v})$
  - $\forall \mathbf{v} \in V$: $1 \cdot \mathbf{v} = \mathbf{v}$
  - $\forall \mathbf{v}, \mathbf{w} \in V, \lambda \in F$: $\lambda(\mathbf{v} + \mathbf{w}) = \lambda\mathbf{v} + \lambda\mathbf{w}$
  - $\forall \mathbf{v} \in V, \lambda, \mu \in F$: $(\lambda + \mu)\mathbf{v} = \lambda\mathbf{v} + \mu\mathbf{v}$



You can combine it...



$$\mathbf{y} = \sum_{i=1}^{n} \lambda_i \mathbf{x}^{(i)}$$

Linear Combinations:
This is basically all you can do.

Algebraically

Notions & Theorems

Definitions (look it up)

- Span ($\mathbf{span}\{\mathbf{x}, \mathbf{y}, \mathbf{z}\}$) – set of linear combinations
- Generating set – set of vectors that span the space
- Basis – minimal set of vectors that span the space
- Dimension – cardinality of basis

Theorems (look it up)

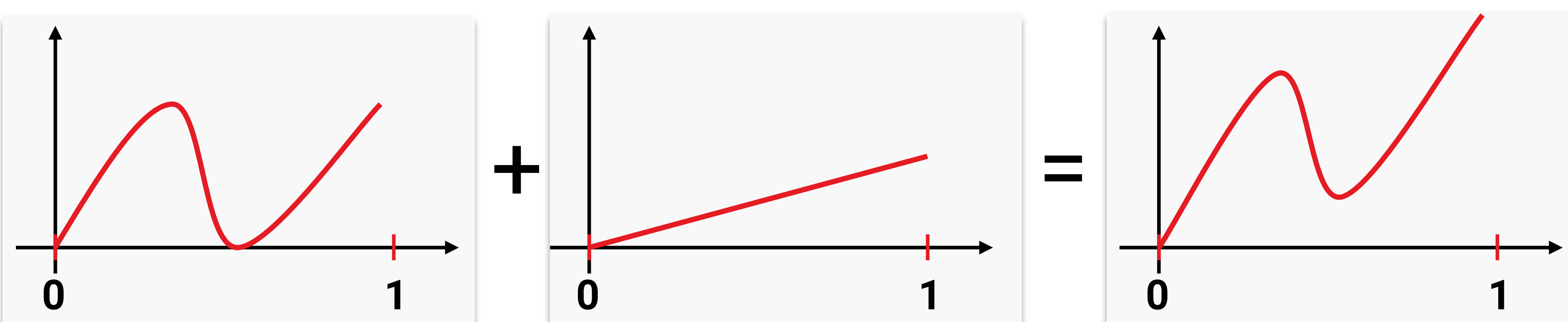
- Every vector space has a basis, cardinality is fixed
- Every finite d-dimensional vector space
  is isomorphic to F^d
  - Proof: Take coordinates in basis, stack up in a vector

Function Spaces

Vector Spaces

Function spaces:

- Space of all functions $f: \mathbb{R} \rightarrow \mathbb{R}$
- Space of all functions $f: [0,1]^2 \rightarrow \mathbb{R}^3$
- etc...



Operations

Adding / Multiplying Functions?

- $(f+g)(x) \;=\; f(x)+g(x)$
- $(\lambda f)(x) \;=\; \lambda f(x)$

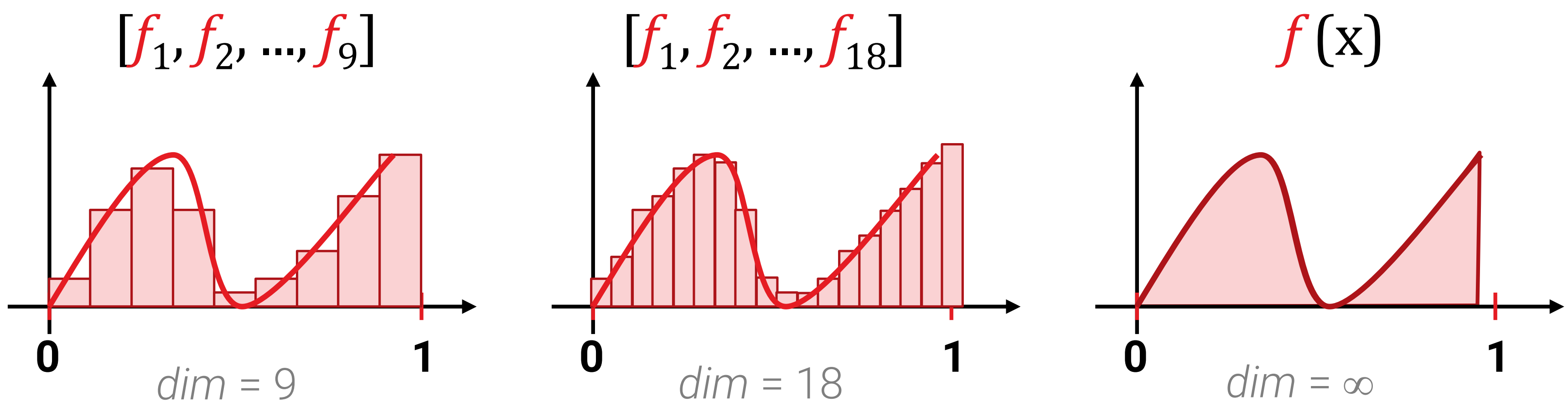
Closed operations?

- Vector spaces:
  - $V_a = \{f:[0,1] \to \mathbb{R} \mid f \text{ continuous}\}$
  - $V_b = \{f:[0,1] \to \mathbb{R} \mid f \text{ differentiable}\}$
  - $V_c = \{f:[0,1] \to \mathbb{R} \mid f \text{ } 2^{\text{nd}} \text{ order polynomial }\}$
- Not a vector space:
  - $V_d = \{f:[0,1] \to \mathbb{R}^{>0}\}$

Function Spaces

Intuition:

- Start with a finite dimensional vector
- Increase sampling density towards infinity
- Real numbers: uncountable amount of dimensions



Example: Image spaces












Example: Image spaces

$$\mathbf{img} = \begin{pmatrix} r_1 \\ g_1 \\ b_1 \\ r_2 \\ \vdots \\ b_d \end{pmatrix}$$

Images

- Vacation photos
- 4000 x 3000 pixel (12 MPixel)
  - RGB images
  - 3 numbers per pixel
- $3 \times 12\text{M} = 36{,}000{,}000$ dimensions
- High-dimensional vector space

Shape Spaces

Examples for Linear Shape Spaces

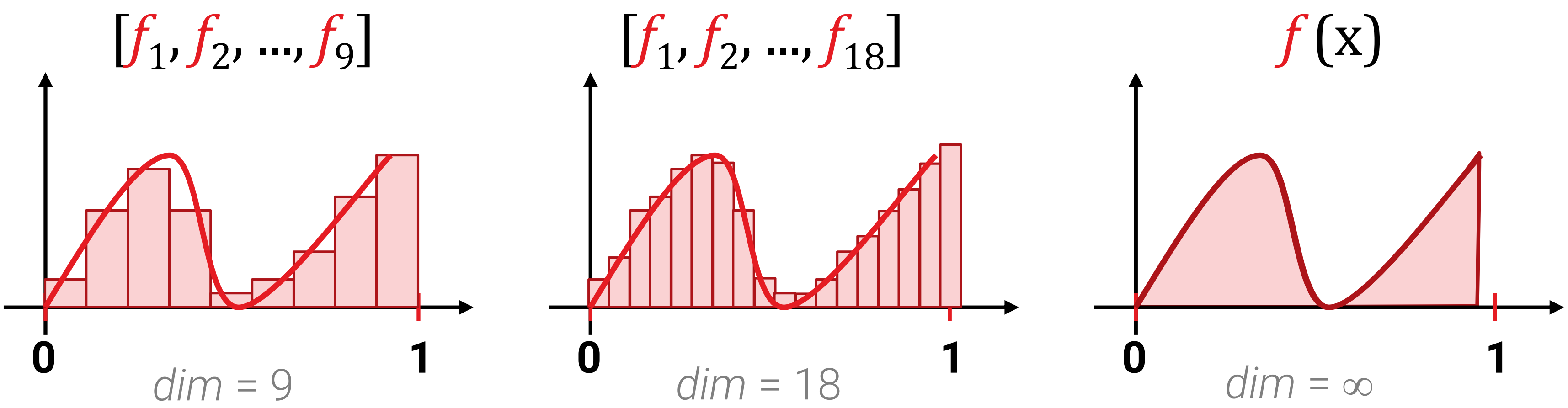
- “Morphable Face Model”
  Volker Blanz & Thomas Vetter
  ACMSiggraph 1999
- https://www.youtube.com/watch?v=jkz-IlIJrig

Back to Generic Function Spaces…

Function Spaces

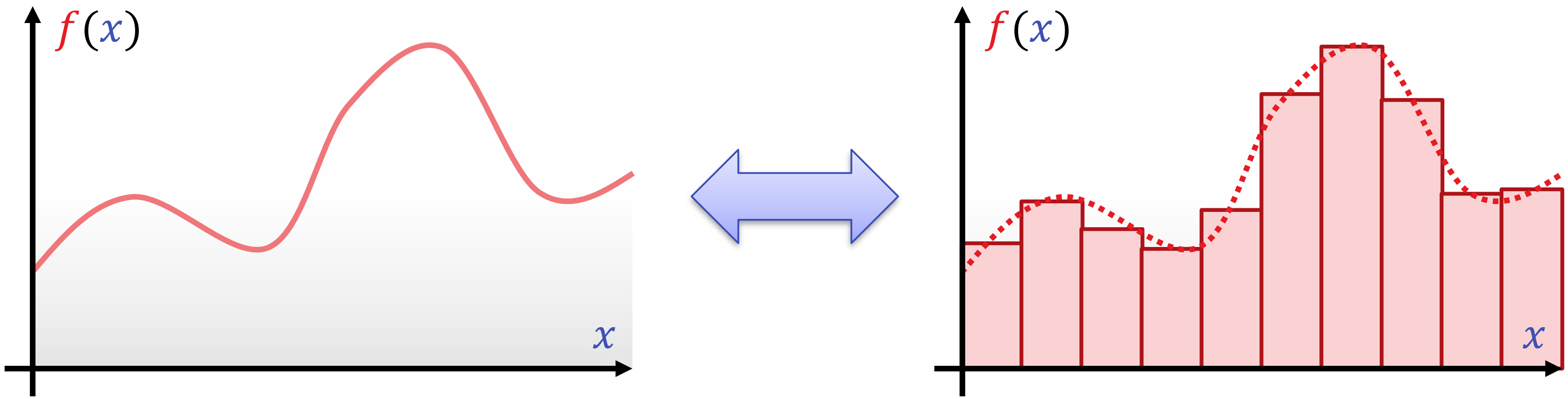
Intuition:

- Start with a finite dimensional vector
- Increase sampling density towards infinity
- Real numbers: uncountable amount of dimensions



Intuition

Function f | Think of this:



spooky, uncountably-infinite thing | array of numbers

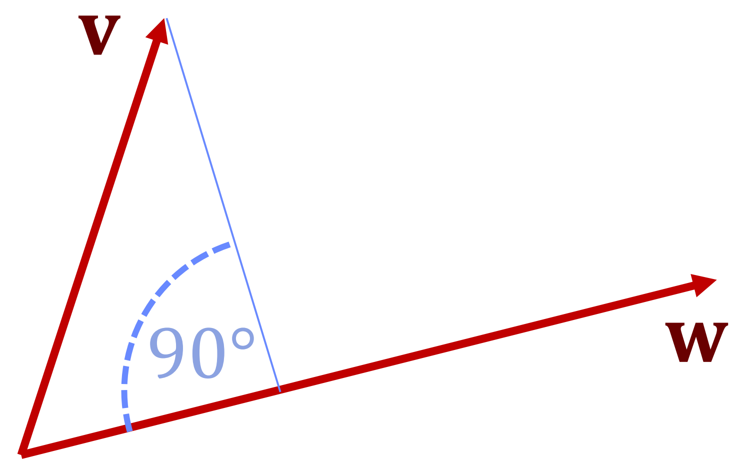
Analogy: Think of functions as *array of numbers*

- Also helps understanding *derivatives*, *integrals*, …
- Subtle differences (pure math lectures)

More Tools:

Angles & Length

Scalar Product



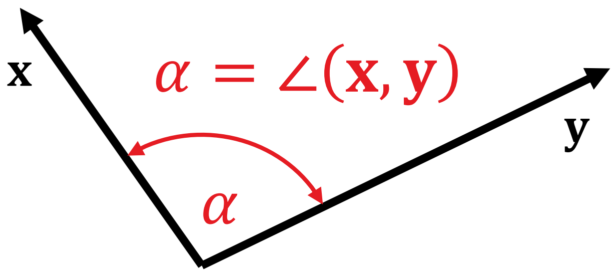
Scalar Product*)

$$\mathbf{v} \cdot \mathbf{w} = \|\mathbf{v}\| \cdot \|\mathbf{w}\| \cdot \cos \angle(\mathbf{v}, \mathbf{w})$$

also: $\langle \mathbf{v}, \mathbf{w} \rangle$

*) also known as *inner product* or *dot-product*

Scalar*) Product



$$\mathbf{x} \cdot \mathbf{y} = \|\mathbf{x}\| \cdot \|\mathbf{y}\| \cdot \cos(\alpha)$$

$$\mathbf{x} \cdot \mathbf{y} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \cdot \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$$

$$= x_1 \cdot y_1 + x_2 \cdot y_2 + x_3 \cdot y_3$$

Scalar*) Product:
measuring angles & length

Algebra:
sum up component product

*) **also known as:** dot product, inner product

Scalar Product on Function Spaces

Scalar products

- For suitable*) functions

$$f, g : \Omega \subset \mathbb{R} \rightarrow \mathbb{R}$$

  the *standard scalar product* is defined as:

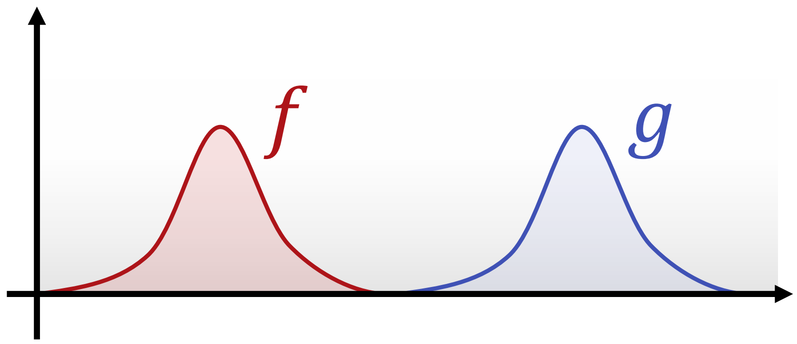
$$f \cdot g = \langle f, g \rangle := \int_{\Omega} f(x) \cdot g(x) \, dx$$

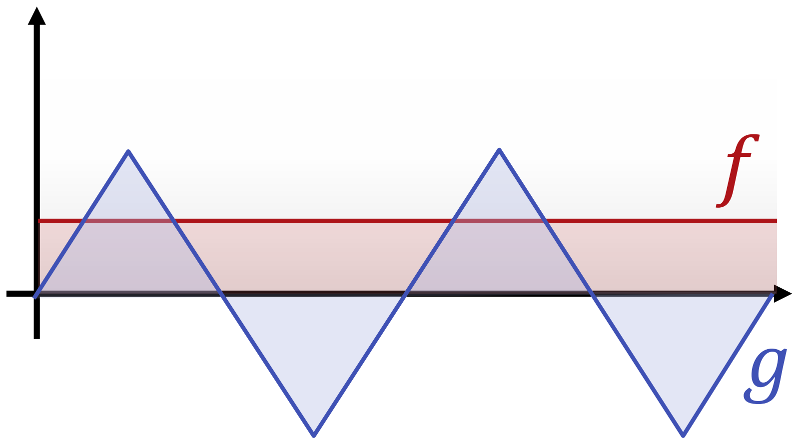
- Measures an *norm* and *angle* in an abstract sense

*) square-integrable

Orthogonal Function

$$\langle f, g \rangle = 0$$





Orthogonal functions

- Do not influence each other in linear combinations.
- Adding one to the other does not change the value in the other ones direction.

Abstract Scalar Product

Abstract scalar product

$$\langle \cdot , \cdot \rangle : V \times V \to F$$

V is an (abstract) vector space,
F is a field (we always use $\mathbb{R}$!)

Scalar product axioms:

$$\forall \mathbf{x}, \mathbf{y}, \mathbf{z} \in V, \lambda \in F:$$

- Symmetry: $\langle \mathbf{x}, \mathbf{y} \rangle = \langle \mathbf{y}, \mathbf{x} \rangle$
- Linearity: $\langle \lambda \mathbf{x}, \mathbf{y} \rangle = \lambda \langle \mathbf{x}, \mathbf{y} \rangle$
  $\langle \mathbf{x} + \mathbf{y}, \mathbf{z} \rangle = \langle \mathbf{x}, \mathbf{z} \rangle + \langle \mathbf{y}, \mathbf{z} \rangle$
- Positive-definiteness: $\langle \mathbf{x}, \mathbf{x} \rangle \geq 0$,
  $[\langle \mathbf{x}, \mathbf{x} \rangle = 0] \Leftrightarrow [\mathbf{x} = \mathbf{0}]$

In Practice…

Finite-dimensional vector spaces

$$\mathbf{x}, \mathbf{y} \in \mathbb{R}^d$$
$$\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x}^{\mathrm{T}} \mathbf{M} \mathbf{y}$$

For a symmetric, positive-definite matrix $\mathbf{M}$

- More on matrices later…

Special case: diagonal matrix

- Function spaces:

$$\langle f, g \rangle := \int_\Omega f(x) \cdot g(x) \cdot \omega(x)\, dx\,, \quad \omega(x) > 0$$

Bases for function spaces

Basis

Examples: bases for function spaces

- Finite dimensional case
  - Polynomials of degree k
  - B-Spline functions over fixed intervals (details soon)
- Countably infinite
  - Set of all polynomials
  - Every linear combination is finite
- Uncountably infinite
  - Set of smooth functions (e.g., $C^0, C^1, \ldots C^\infty$)
  - Set of square integrable functions (L_2)
  - Hard to construct a basis

Schauder Basis

Schauder Basis

- Series representation of vectors
- Important for function spaces

Definition: Schauder-Basis of V

- Sequence of basis vectors $\mathbf{b}_1, \mathbf{b}_2, \ldots \in V$
- For every $\mathbf{v} \in V$, there is a unique sequence $\lambda_1, \lambda_2, \ldots \in F$ such that

$$\lim_{n \to \infty} \left\| \mathbf{v} - \sum_{i=1}^{n} \lambda_i \mathbf{b}_i \right\| = 0$$

Schauder Basis

Function spaces L^p:

- $\langle f, g \rangle := \int_\Omega f(x) g(x) dx$, $\quad \|f\|_2 := \sqrt{\langle f, f \rangle}$
- L^2 space of square-integrable functions: Integral exists
- Analogous L^p:

$$\|f\|_p := \left(\int_\Omega |f(x)|^p dx \right)^{\frac{1}{p}}$$

Schauder Basis

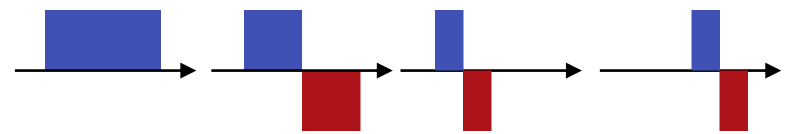
Schauder-Bases for L^p

- Fourier basis with $\Omega = [0,2\pi)$

$$B = \left\{ \frac{1}{2}\sqrt{2}\,\sin 2\pi k x\,, \frac{1}{2}\sqrt{2}\,\cos 2\pi k x \,\middle|\, k \in \mathbb{N} \right\}$$

(excluding k=0 here)

- Haar basis for fixed intervals



Sequence space as substitute

- l_2 with norm $\|(\lambda_1, \lambda_2, \ldots)^T\| := \sqrt{|\lambda_1|^2 + |\lambda_2|^2 + \cdots}$
- L^2 on $[0,2\pi]$
- L^2 can be approximated (to arbitrary precision) with l_2

“Standard” Setting

We usually consider the Hilbert Space L^2

Hilbert space

- Scalar product $\langle f, g \rangle$
- Norm $\|f\| := \sqrt{\langle f, f \rangle}$
- Complete space (convergent “Cauchy” series do have a limit)

“Standard” Setting

We usually consider the Hilbert Space L^2

Space L^2

- Functions $f: \Omega \to \mathbb{R}$ with domain $\Omega \subseteq \mathbb{R}^d$, square integrable ($\langle f, g \rangle$ exists for all $f, g \in L^2$)
- $\langle f, g \rangle \coloneqq \int_{\Omega} f(x) \cdot g(x)\, dx$ (Lesbeque integral)
- For $\Omega = [0,2\pi)^d$, the Fourier-Basis is a suitable Schauder-Basis
  - Isometry between L^2 and ℓ_2 (= square-summable sequences, with standard scalar product w/infinite sum)

“Standard” Setting

Really?

If we perform numerical computations

- We just use a finite-dimensional representation
- An array is a good starting point

Let’s look at this in more detail…

…in one of the later sections!